Elementary maths for GMT

Linear Algebra

Part 1: Vectors, Representations

Algebra and Linear Algebra

- Algebra: numbers and operations on numbers
  - $2 + 3 = 5$
  - $3 \times 7 = 21$

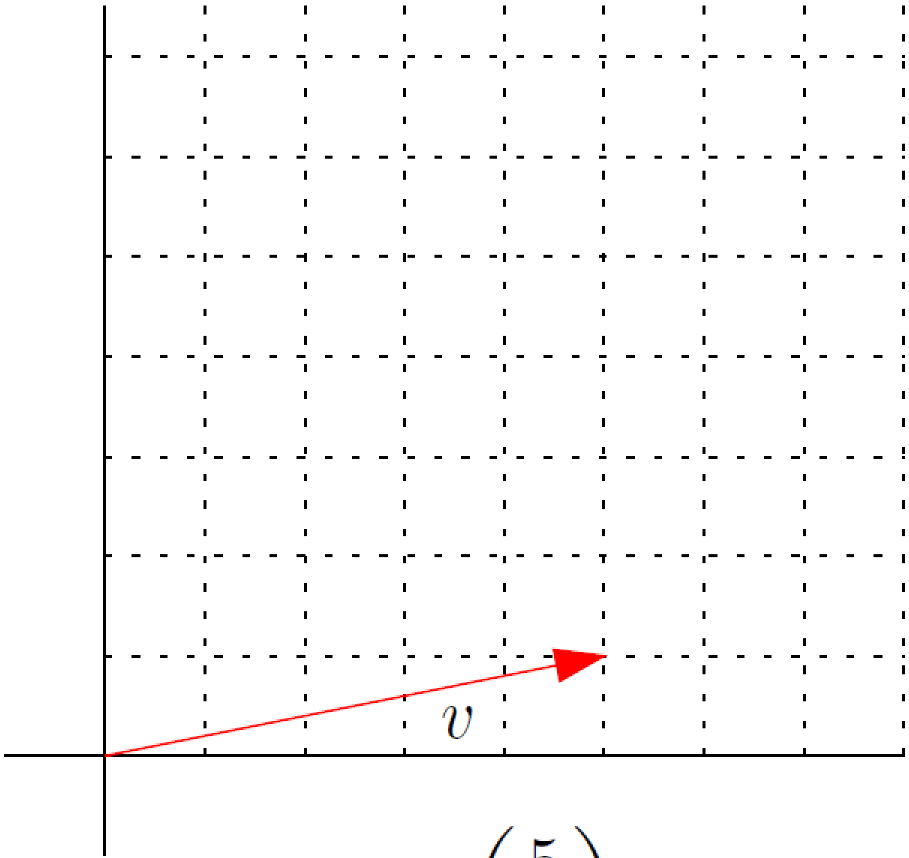
- Linear Algebra: tuples, triples ... of numbers and operations on them
  - Assists in geometric computations

Vectors: definition

- A vector in $\mathbb{R}^d$ is an ordered d-tuple $v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{pmatrix}$
- In $\mathbb{R}^3$, for example: $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$
  - (or $\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$, or $\begin{pmatrix} x_v \\ y_v \\ z_v \end{pmatrix}$, or (v_1, v_2, v_3), or ...)

Vectors: algebraic interpretation

- A 2D vector $\binom{x_v}{y_v}$ can be seen as the point (x_v, y_v) in the Cartesian plane



$$v = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

Vectors: notation

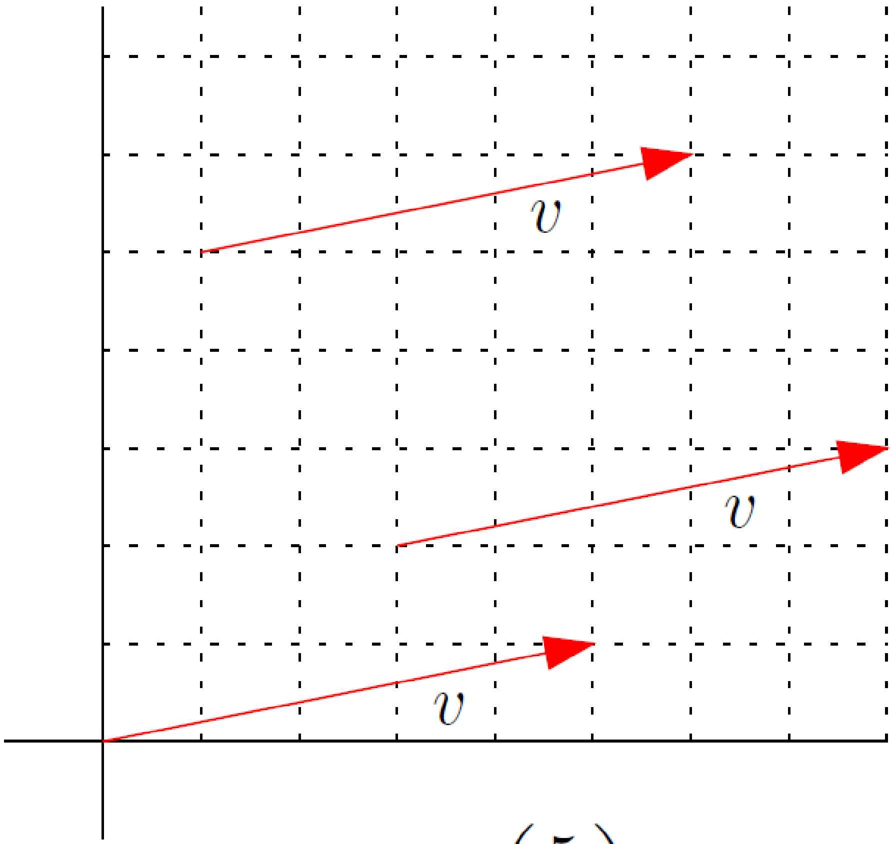
- A 2D vector should be denoted as $\binom{x_v}{y_v}$ or as $(x_v, y_v)^T$
  - The T in the exponent stands for *transposed*

- A 2D point should be denoted as (x_v, y_v)

- Be aware of misused notation (mostly point notation used for a vector)

Vectors: geometric interpretation

- A 2D vector $\binom{x_v}{y_v}$ can be seen as an offset from the origin
- Such an offset (arrow) can be translated



$$v = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

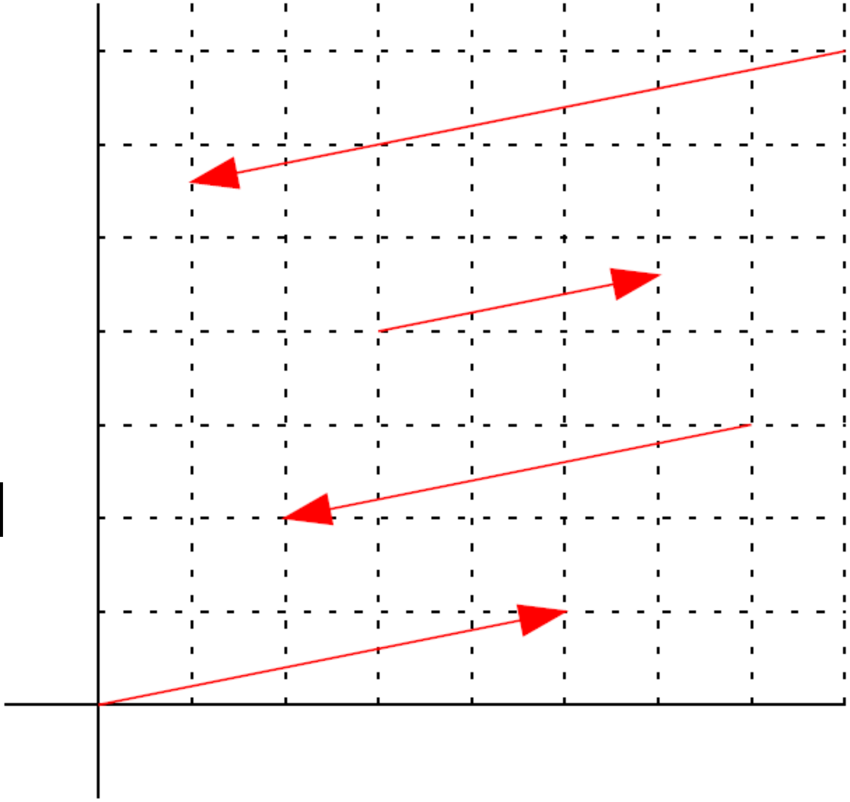
Vectors: length and scalar multiple

- The Euclidean length of a d-dimensional vector v is $\|v\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_d^2}$
- A scalar multiple of a d-dimensional vector v is $\lambda v = (\lambda v_1, \lambda v_2, \ldots, \lambda v_d)^T$
  - Note that v and λv have the same direction or opposite directions

Parallel vectors

- Two vectors v_1 and v_2 are parallel if one is a scalar multiple of the other, *i.e.* there is a $\lambda \neq 0$ such that $v_2 = \lambda v_1$

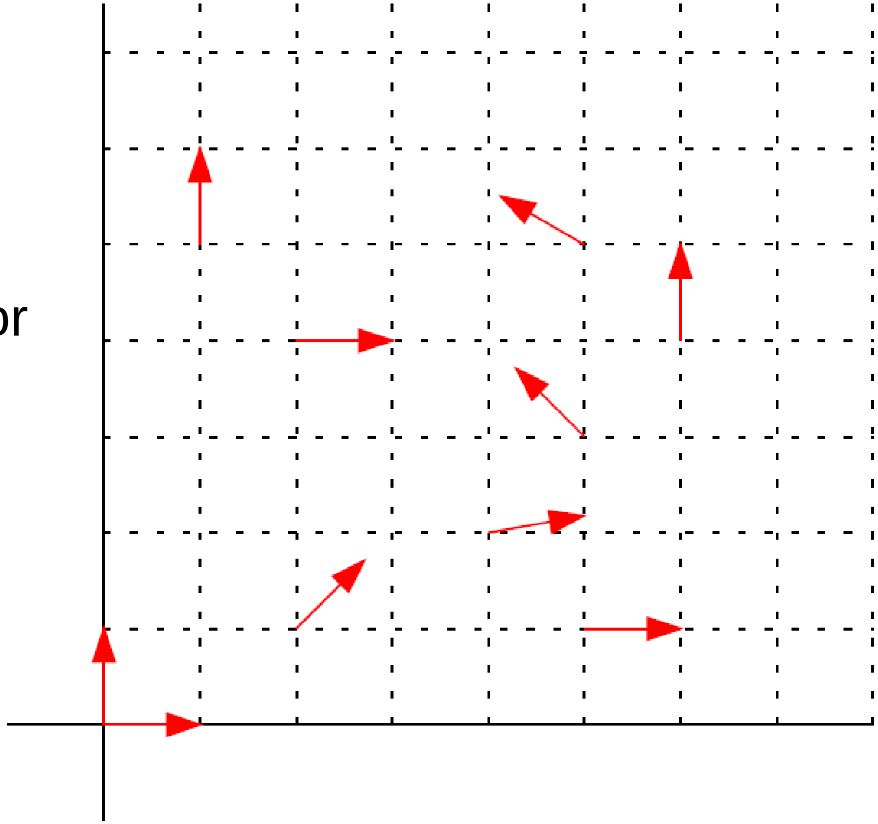
- Note that if one of the vectors is the null vector, then the vectors are considered neither parallel nor not parallel



Unit vectors

- A vector v is a unit vector if $\|v\| = 1$

- Normalization
  - Questions
    - Given an arbitrary vector v, how do we find a unit vector parallel to v?
    - Can every vector be normalized?

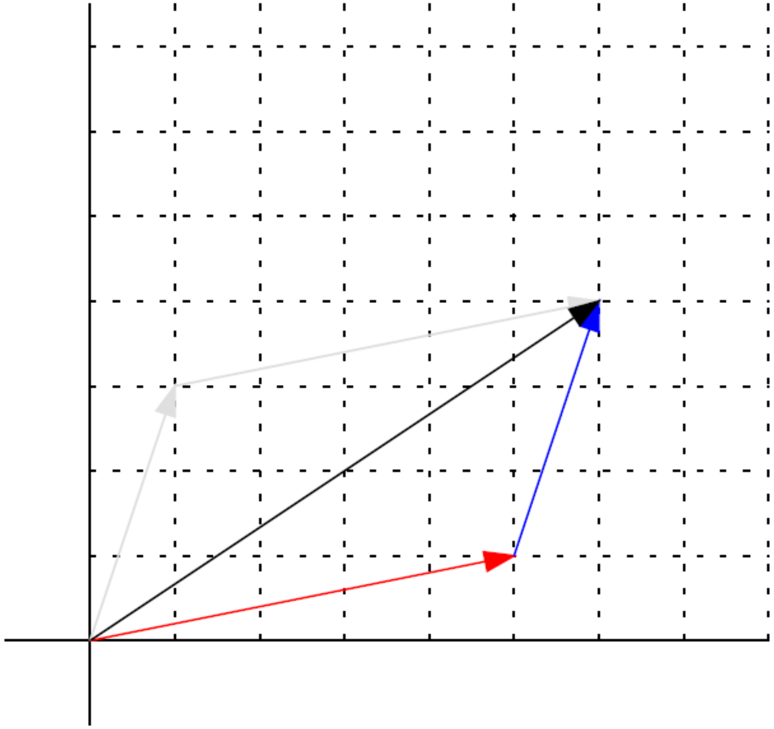


Addition of vectors

- Given two vectors in $\mathbb{R}^d$,
  $v = (v_1, v_2, \dots, v_d)^T$ and
  $w = (w_1, w_2, \dots, w_d)^T$

  their sum is defined as

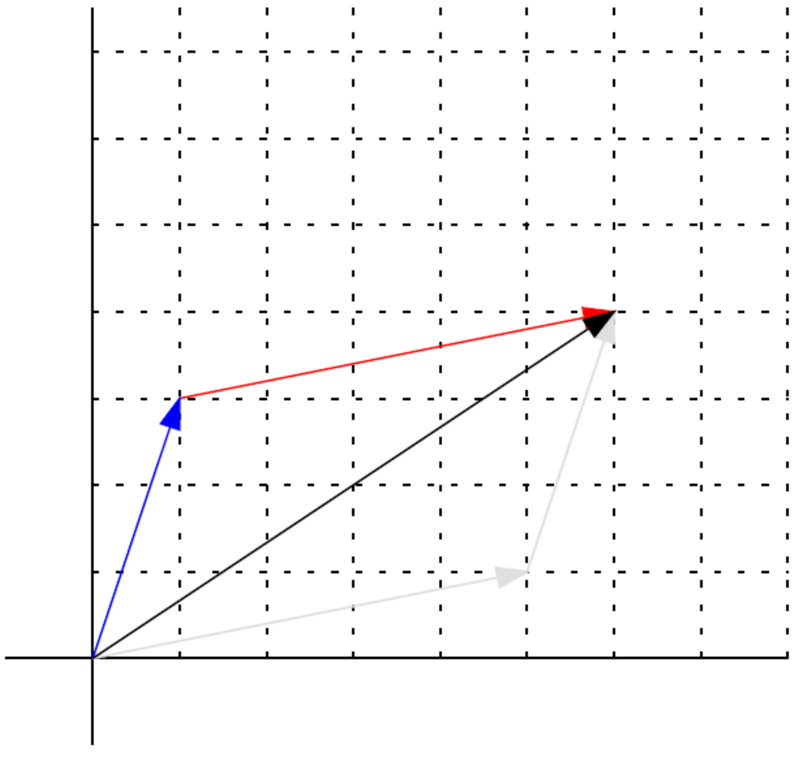
  $v + w =$
  $(v_1 + w_1, v_2 + w_2, \dots, v_d + w_d)^T$



- Q: How would subtraction be defined?

Addition of vectors

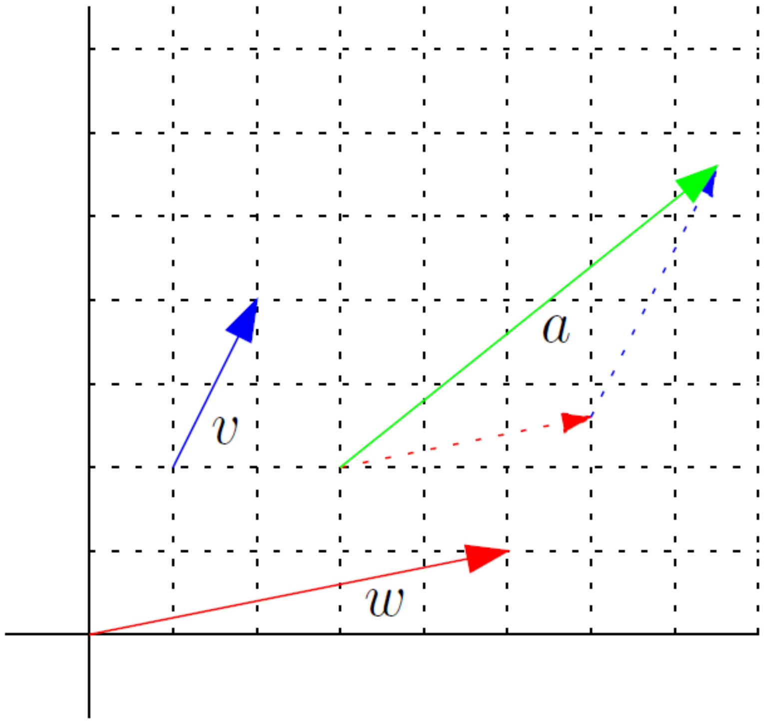
- Addition of vectors is commutative as it can be seen easily from the geometric interpretation



- Q: show algebraically that vector addition is commutative
- Q: what is the relation between $||v||$, $||w||$, and $||v + w||$?

Bases in 2D

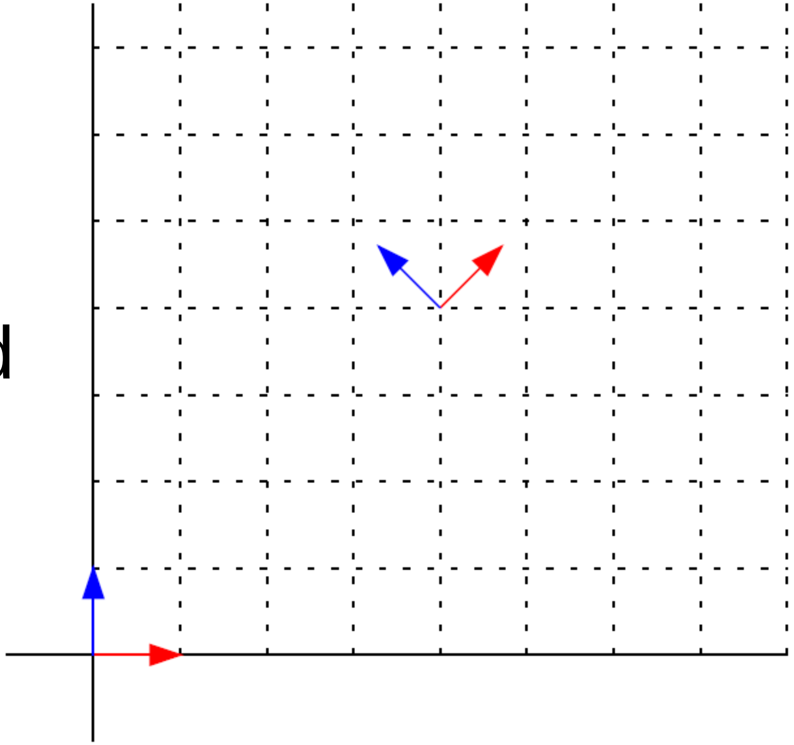
- A 2D vector can be expressed as a combination of any pair of non-parallel vectors
  - For instance, in the figure, $a = 1.5v + 0.6w$
- Such a pair is called linearly independent, and forms a 2D basis
- The extension to higher dimensions is straightforward



Orthonormal basis in 2D

- Two vectors form an orthonormal basis in 2D if (1) they are orthogonal to each other, and (2) they are unit vectors

- The advantage of an orthonormal basis is that lengths of vectors, expressed in the basis, are easy to calculate



The null vector

- The null vector $\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ is special
  - It acts as the zero for addition of vectors
  - It is the only vector that has length zero
  - It is the only vector that does not have a direction
  - It can not be used as a base vector

Dot product

- For two vectors $v, w \in \mathbb{R}^d$, the dot product is defined as

$$v \cdot w = v_1 w_1 + v_2 w_2 + \cdots + v_d w_d,$$

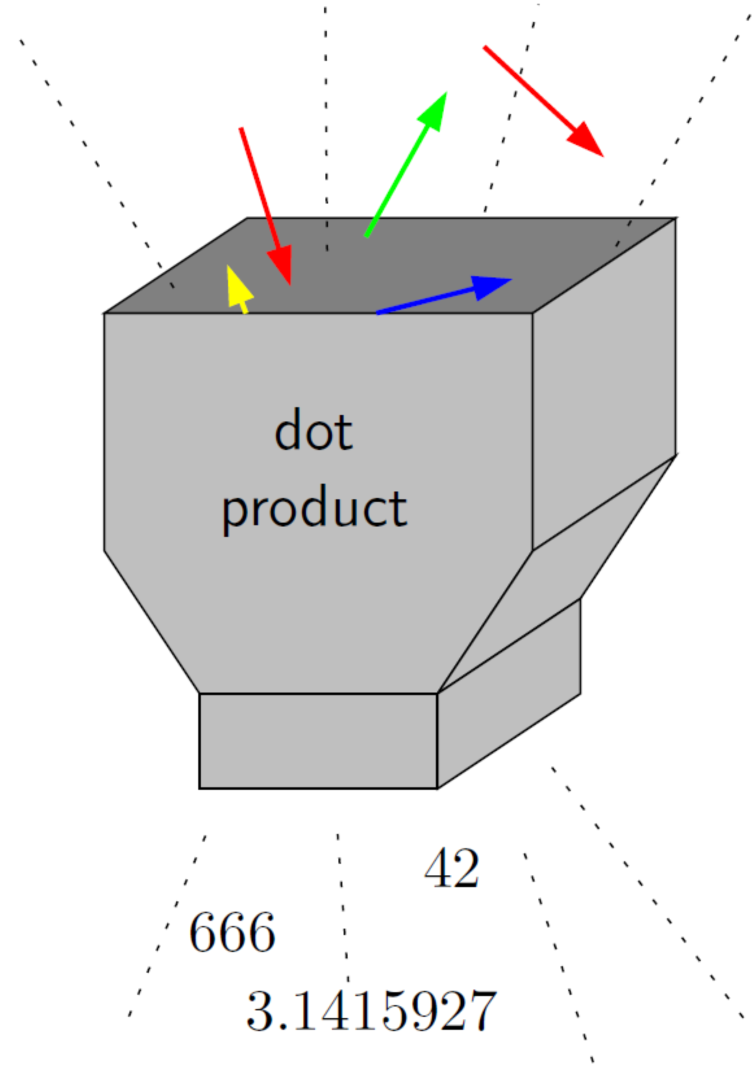
or

$$v \cdot w = \sum_{i=1}^{d} v_i w_i$$

- We have $\cos\theta = \frac{v \cdot w}{\|v\| \|w\|}$, where θ is the angle between the two vectors
- Note that the dot product is also called inner product or scalar product and that the result of the operation is a scalar (not a vector)

Dot product

- Questions
  - What is the inner product of an arbitrary unit vector with itself?
  - What do we know if for two vectors v and w we have that $v \cdot w = 0$?



Cross product

- For two vectors $v, w \in \mathbb{R}^3$, the cross product is defined as

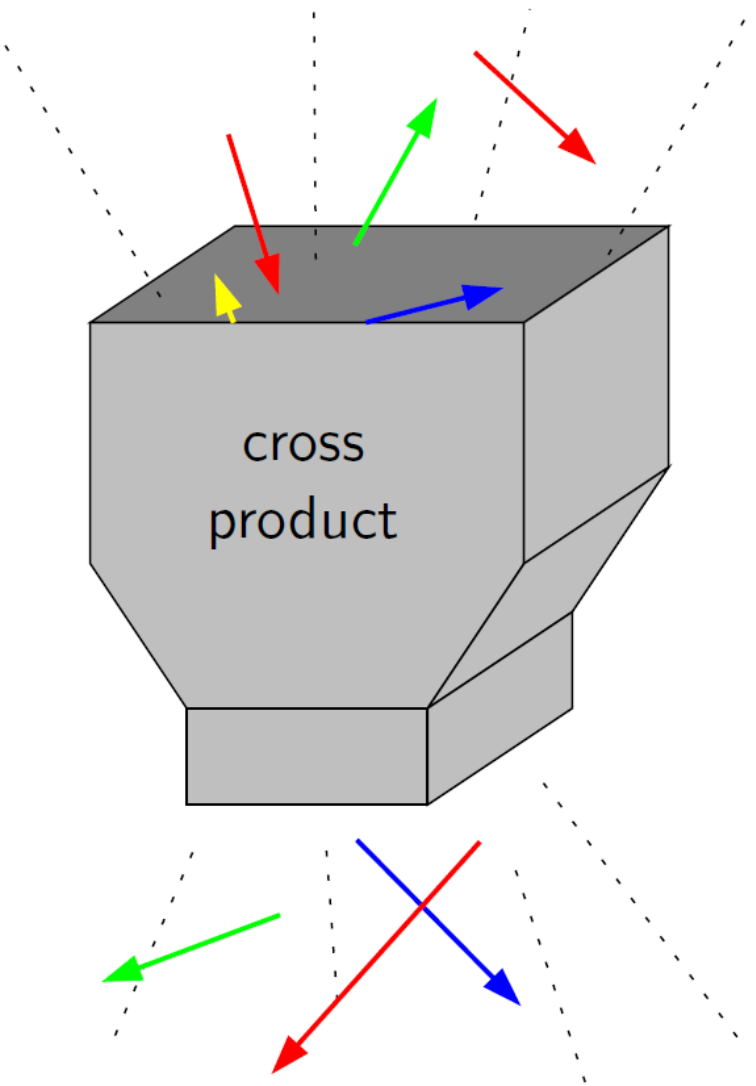
$$v \times w = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}$$

- Q: Show that $v \times w$ is orthogonal to both v and w

- We have that $||v \times w|| = ||v|| \, ||w|| \sin\theta$, where θ is the angle between v and w
- Note that the result of the operation is a vector

Cross product

- Questions
  - It is possible or necessary that v and w are orthogonal to form $v \times w$?
  - What is $v \times w$ if v and w are parallel?



Products and null vector

- Questions
  - What is the dot product of a vector and the null vector?
  - What is the cross product of a vector and the null vector?

Bases in 3D

- You need three vectors to form a basis in 3D
- If u, v, and w form a basis, then any vector a in 3D can be expressed as

$$a = \mu u + \lambda v + \rho w$$

where μ, λ, and ρ are scalars

- Q: Let u, v, and w be three vectors (no one is the null vector). Suppose that u and v are not parallel, u and w are not parallel, and v and w are not parallel. Do u, v, and w always form a basis?

Linear dependence in 3D

- If for three vectors u, v, and w in 3D (no null vectors), we have $w = \mu u + \lambda v$

  where μ and λ are scalars, then u, v, and w are linearly dependent
- If such μ and λ do not exist, then they are linearly independent
- Any three linearly independent vectors in 3D form a 3D basis

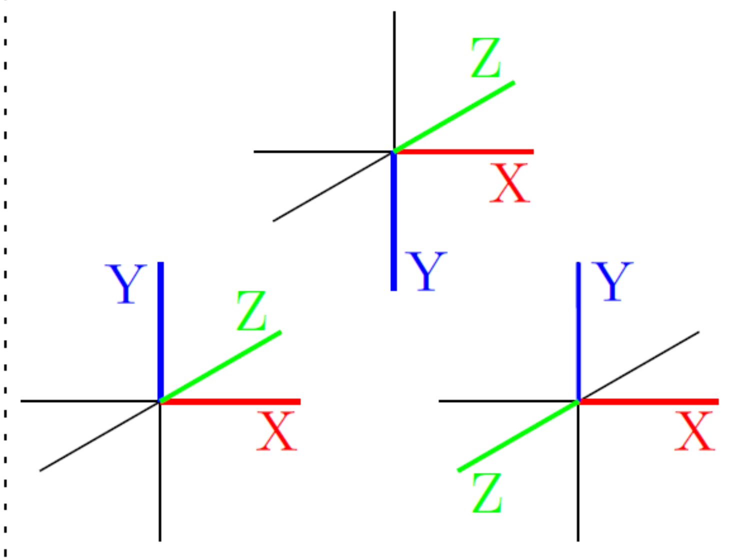
Orthonormal 3D bases

- Three vectors form an orthonormal basis in 3D if (1) each pair of them is orthogonal, and (2) they are unit vectors

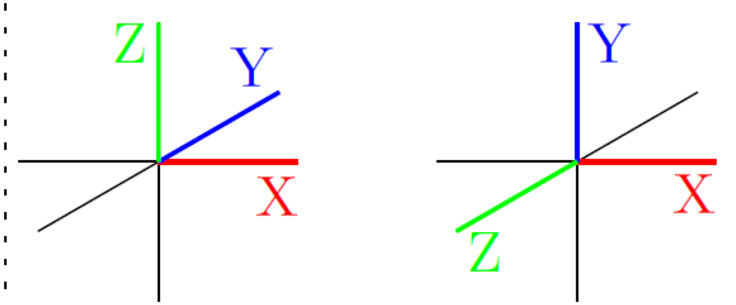
- Questions
  - What would you do to test if three 3D vectors form an orthonormal basis?
  - Suppose that two vectors u and v in 3D are orthogonal, and they are unit vectors. Let w be the cross-product of u and v. What can you say about u, v, and w (do they form an orthonormal basis)?

Left- and right-handed systems

- Coordinate systems in 3D come in two flavors: left-handed and right-handed
- There are arguments for both systems for
  - The global system
  - The camera system
  - Objects systems



Camera coordinate system



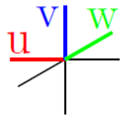
World coordinate system

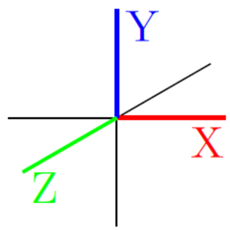
Coordinate transformations

- A frequent operation in graphics is the change from one coordinate system (*e.g.* the (u, v, w) camera system) to another (*e.g.* the (x, y, z) global system)
- Having orthonormal bases

  for both systems makes the

  transformations simpler

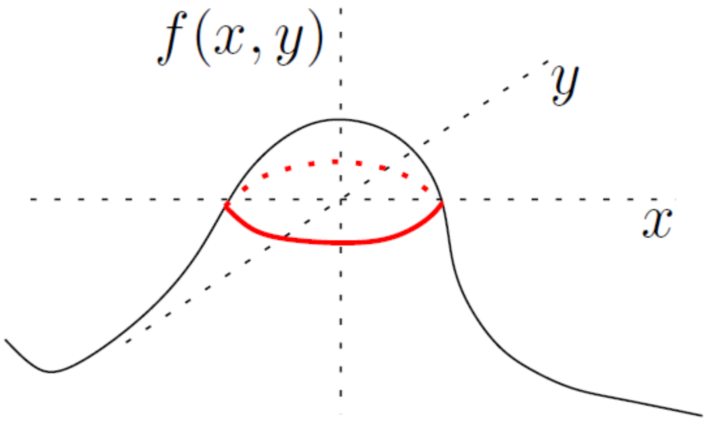


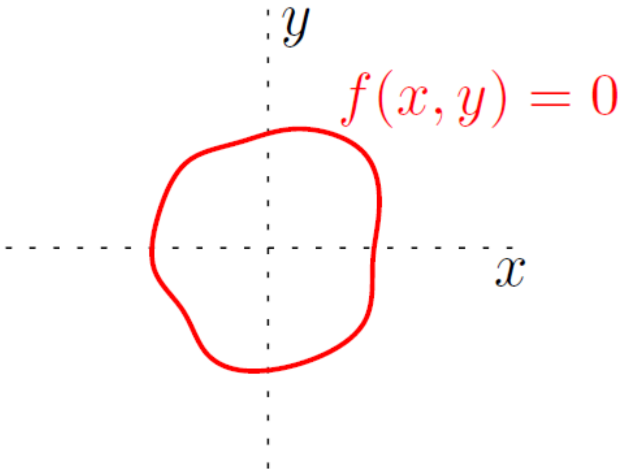


2D implicit curves

- An implicit curve in 2D has the form $f(x, y) = 0$

- f maps two-dimensional points to a real value; the points for which this value is 0 are on the curve, while other points are not on the curve

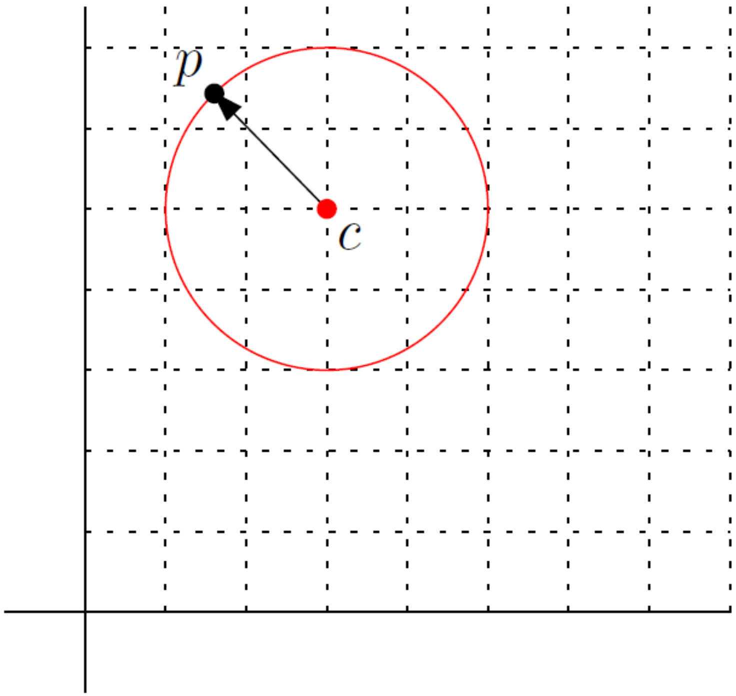




Implicit representation of circles

- The implicit representation of a 2D circle with center c and radius r is
$$(x - x_c)^2 + (y - y_c)^2 - r^2 = 0$$
- So for any point p that lies on the circle, we have

$(p - c) \cdot (p - c) - r^2 = 0$, so
$\|p - c\|^2 - r^2 = 0$, which gives
$\|p - c\| = r$



Implicit representation of lines

- A well-known representation of lines is the slope-intercept form

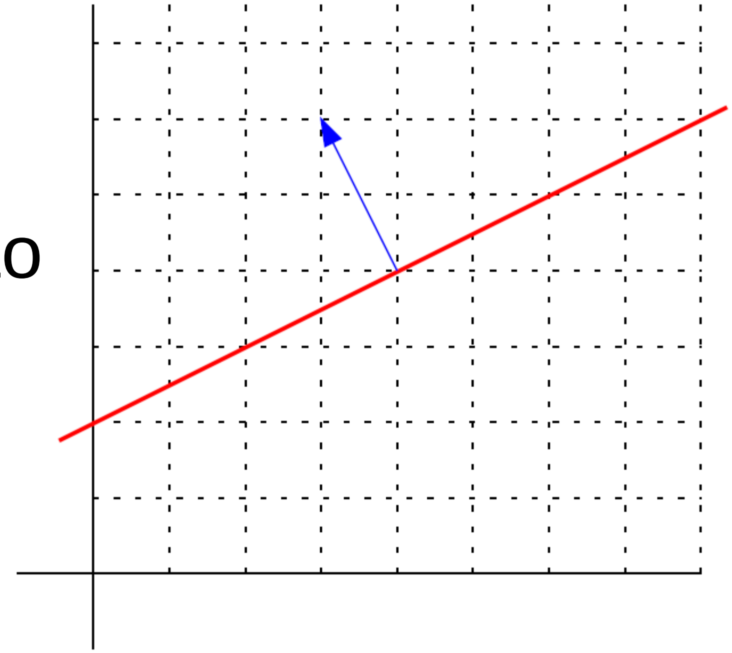
$$y = ax + b$$

- This can easily be converted to

$$-ax + y - b = 0$$

- If $b = 0$, the line intersects the origin, and we have

$$n \cdot p = 0\text{, with } p = \begin{pmatrix} x \\ y \end{pmatrix} \text{ and } n = \begin{pmatrix} -a \\ 1 \end{pmatrix}$$



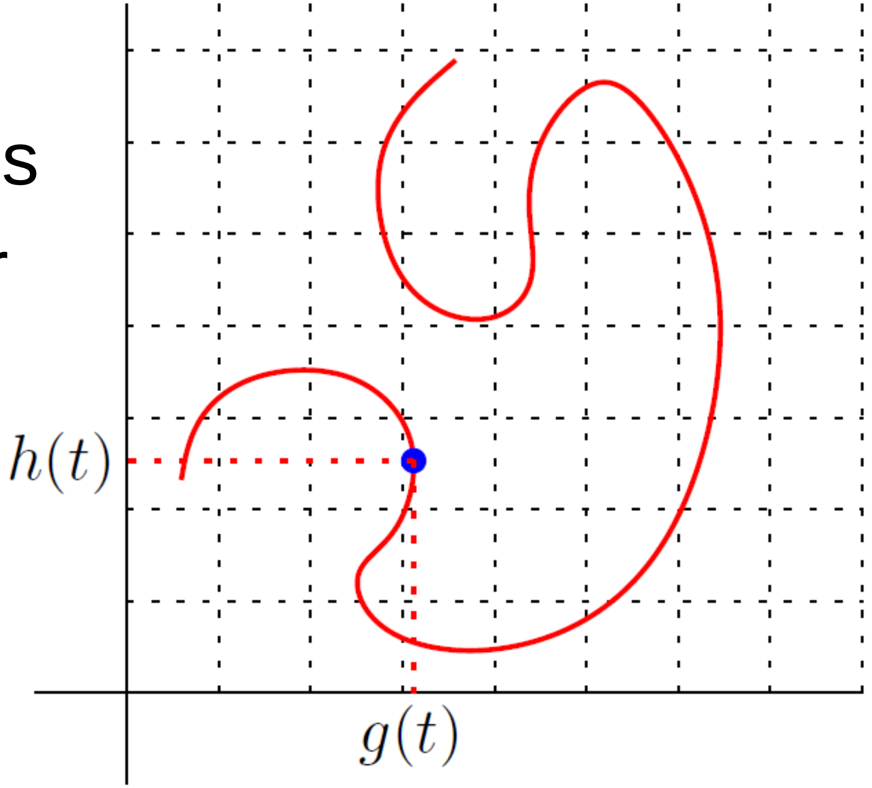
- Q: What if the line does **not** intersect the origin?

2D parametric curves

- A parametric curve is controlled by a single parameter, and has the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} g(t) \\ h(t) \end{pmatrix}$$

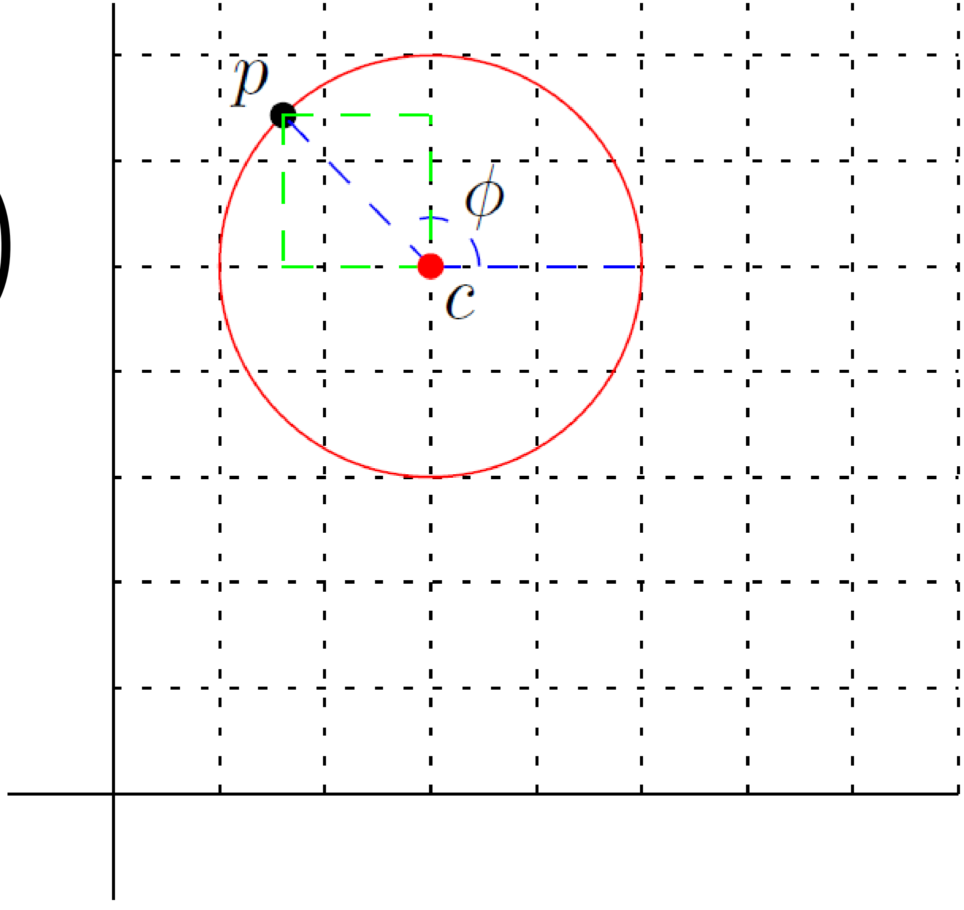
- Parametric representations have some advantages over functions, even if a function would suffice to represent the curve



Parametric equation of a circle

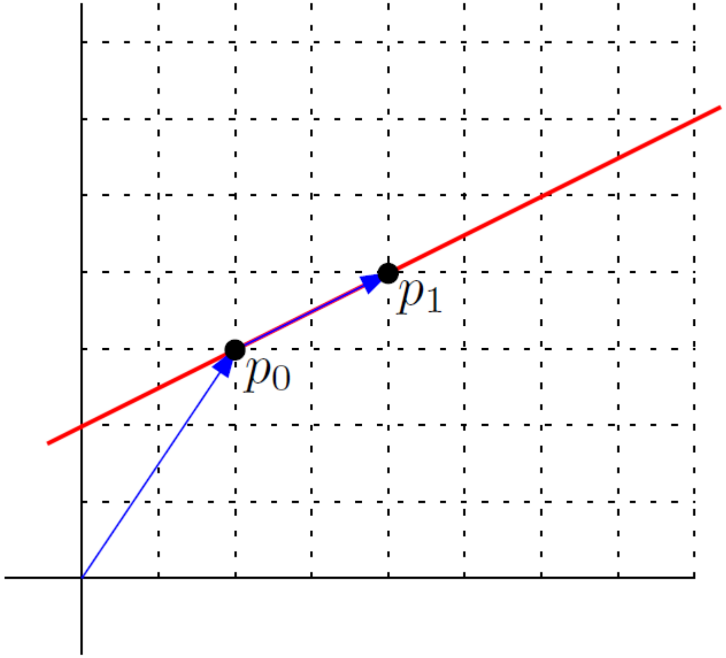
- The parametric equation of a 2D circle with center c and radius r is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_c + r\cos\phi \\ y_c + r\sin\phi \end{pmatrix}$$



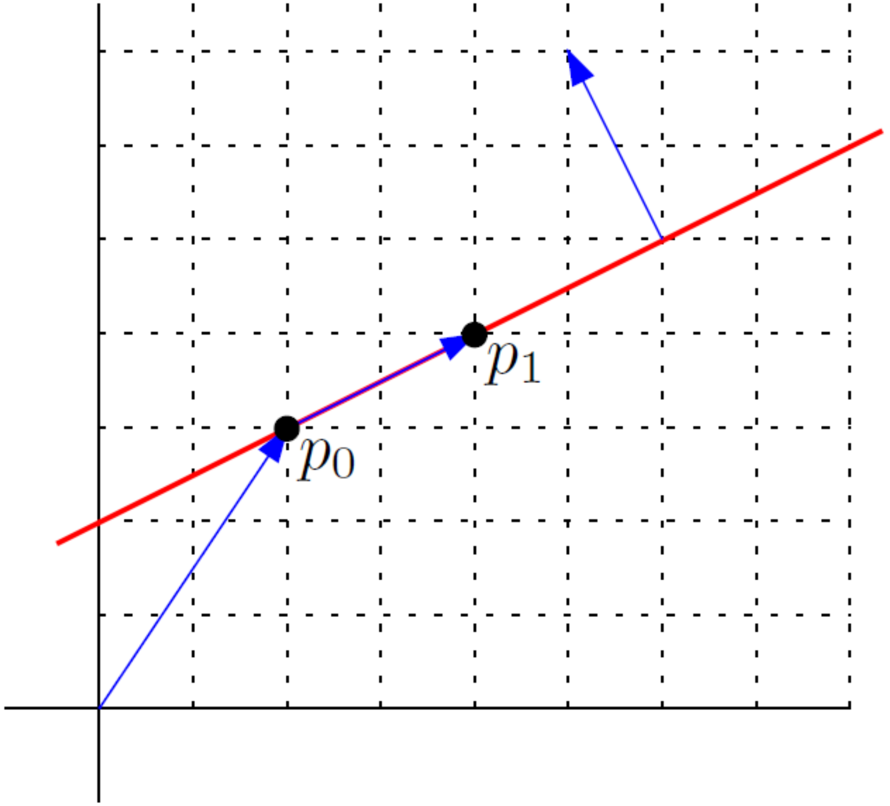
Parametric equation of a line

- The parametric equation of a line through the points p_0 and p_1 is
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_{p_0} + t(x_{p_1} - x_{p_0}) \\ y_{p_0} + t(y_{p_1} - y_{p_0}) \end{pmatrix}$$
- This can alternatively be written as
$$p(t) = p_0 + t(p_1 - p_0)$$



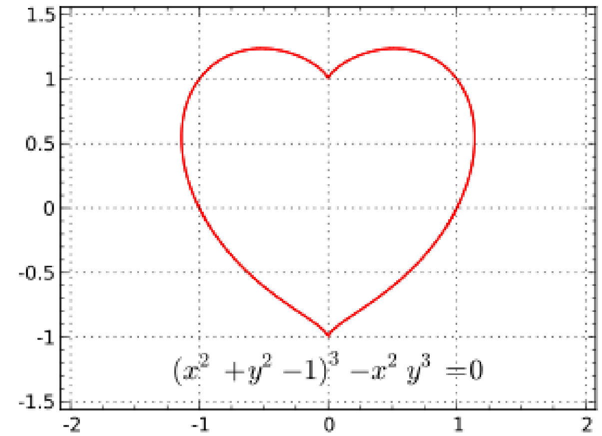
Conversion between representations

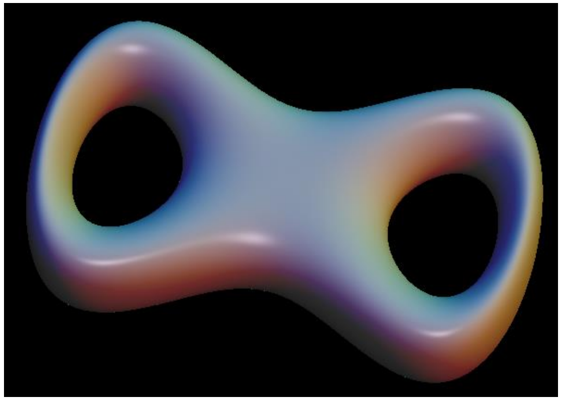
- It is convenient to be able to convert a parametric equation of a line into an implicit equation, and vice versa
- Q: How do we do that?



Implicit surfaces: from 2D to 3D

- Recall that an implicit curve has the form $f(x, y) = 0$
- The 3D generalization is an implicit surface with a similar form $f(x, y, z) = 0$
- Fun project: try to draw the 4D image of the graph of such function

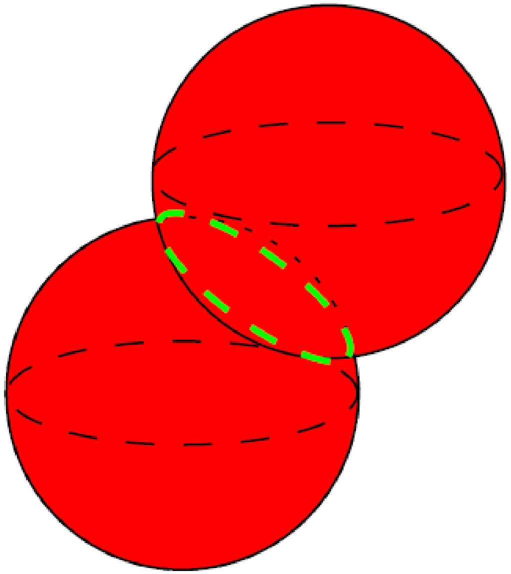




$$(x^2 \times (1 - x^2) - y^2)^2 + \frac{z^2}{2} - \frac{1}{40}\left(1 + (x^2 + y^2 + z^2)\right) = 0$$

Implicit one-dimensional curves in 3D?

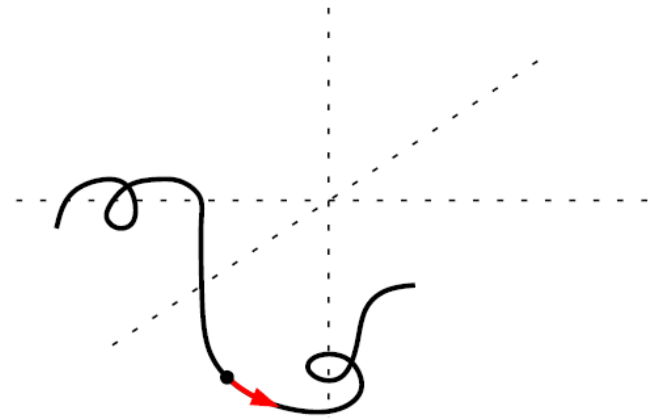
- Cooking up an implicit function for a one-dimensional thingy in 3D is in general not possible; such thingies are degenerate surfaces
  - For example, $x^2 + y^2 = 0$ is a cylinder with radius 0: the Z-axis
- More complex curves can be described as the intersection of two or more implicit surfaces



Parametric curves and surfaces

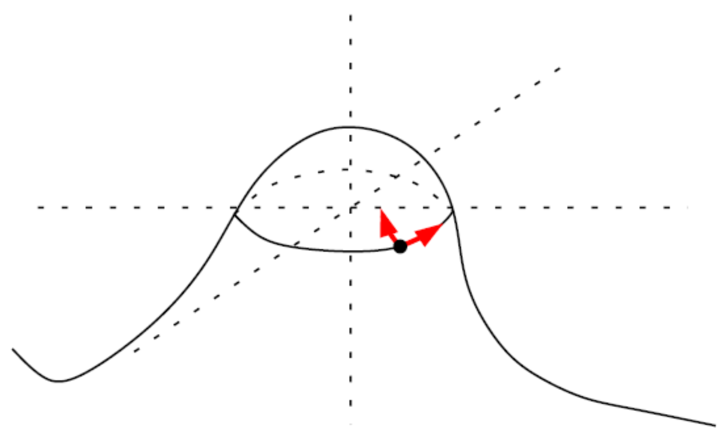
- As opposed to implicit curves, it is possible to specify parametric curves in 3D

$$x = f(t),$$
$$y = g(t),$$
$$z = h(t)$$



- Parametric surfaces depend on two parameters

$$x = f(u, v),$$
$$y = g(u, v),$$
$$z = h(u, v)$$



Implicit spheres

- The sphere equation is given by:

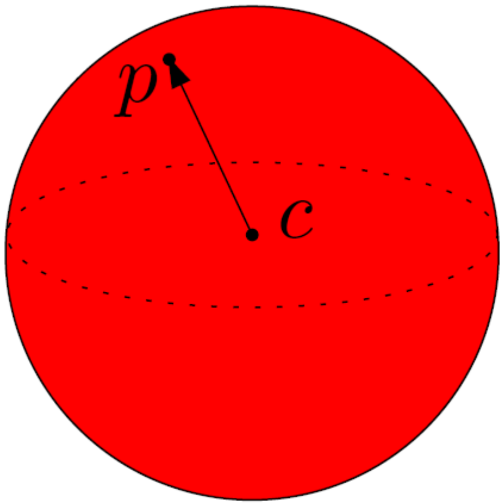
$$(x - c_x)^2 + (y - c_y)^2 + (z - c_z)^2 - r^2 = 0$$

- Just as in the circle case, this can be written in dot product form for any point p on the sphere

$(p - c) \cdot (p - c) - r^2 = 0$, so

$\|p - c\|^2 - r^2 = 0$, which gives

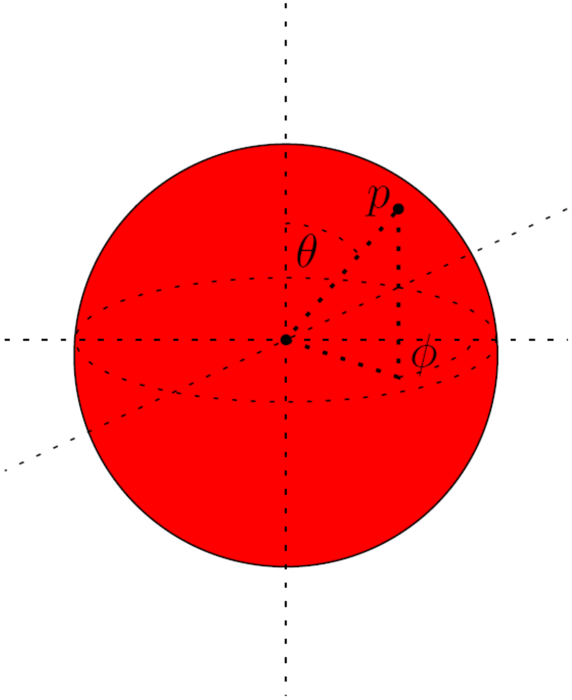
$\|p - c\| = r$



Parametric spheres

- Spheres can also be represented parametrically
  - For example, a sphere with radius r centered at the origin has the equation

$$x = r\cos\phi\sin\theta\,,$$
$$y = r\sin\phi\sin\theta\,,$$
$$z = r\cos\theta$$



- Q: What would the equation for a sphere with radius r centered at $c = (c_x, c_y, c_z)$ be?

Parametric spheres

$$x = r\cos\phi\sin\theta\,,$$
$$y = r\sin\phi\sin\theta\,,$$
$$z = r\cos\theta$$

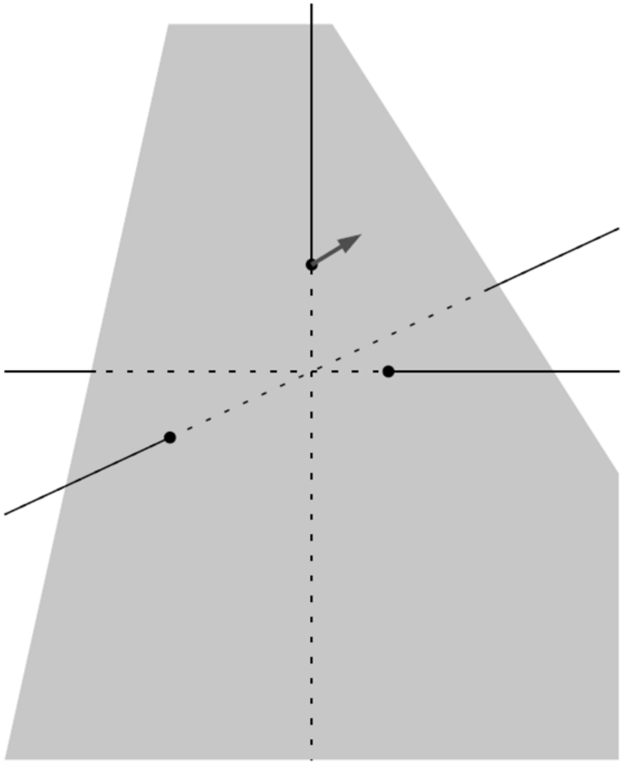
- The parametric representation of a sphere looks much more inconvenient than the implicit equation
- However, when we have to do texture mapping, the parametric representation turns out to be quite convenient

Implicit planes

- The implicit equation for a plane in 3D looks a lot like the equation for a line in 2D

$$ax + by + cz - d = 0$$

- Here, $(a, b, c)^T$ is a normal vector of the plane

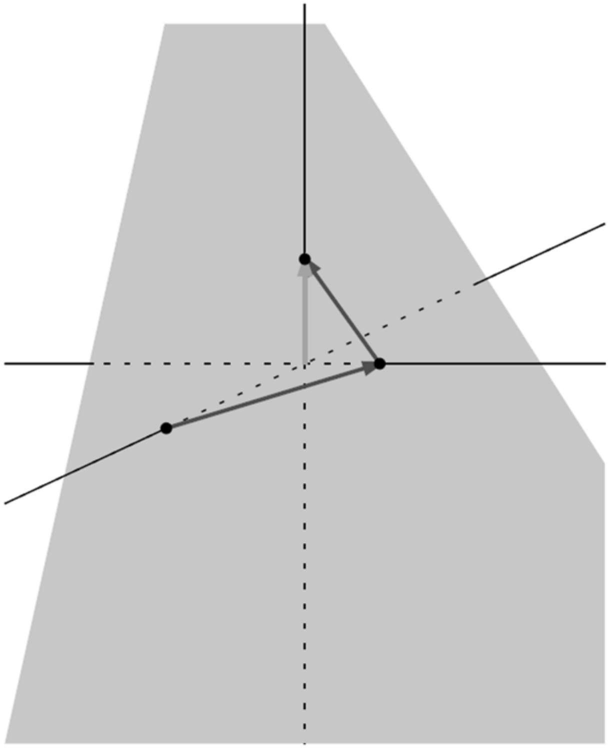


- Q: What is the meaning of d?

Parametric planes

- Planes can also be described parametrically
- Instead of one direction vector (as for lines), we need two

$$(x, y, z) = (x_p, y_p, z_p) + s(x_v, y_v, z_v)^T + t(x_w, y_w, z_w)^T$$



Implicit and parametric planes

- Implicit equation:

$$ax + by + cz - d = 0$$

- Parametric equation:

$$(x, y, z) = \left(x_p, y_p, z_p\right) + s(x_v, y_v, z_v)^T + t(x_w, y_w, z_w)^T$$

- Questions
  - Is an implicit description of a plane in 3D unique?
  - Is a parametric description of a plane in 3D unique?